Identification and Estimation of Sequential Games of Incomplete Information with Multiple Equilibria

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Introduction

A sequential game with finite players

- Sequential Entry Games, Bargaining games between groups
- Different order of actions leads to different strategic behavior
- Researchers may not observe the "*true*" order of actions

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  - The former player may not know a lot about the later player's payoff

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- Researchers may not observe the "*true*" order of actions
- *Incomplete information*
  - Player knows her own payoff, not the rival's payoff
  - The former player may not know a lot about the later player's payoff
- *Order of actions / Multiple equilibria*
  - Observable to players, Unobservable to researchers
  - The misspecified equilibrium type / order of actions may lead to inaccurate estimates on structural parameters

Contribution

1. First theoretical approach to empirical models for sequential games considering:
   1.1 Incomplete information game
   1.2 Multiple Perfect Bayesian Nash Equilibria
   1.3 Unknown distribution of the order of actions
2. Semiparametric identification, estimation, and inference on structural parameters:
   2.1 Payoff function parameters
   2.2 Equilibrium selection mechanism
   2.3 Distribution of the order of actions
3. Empirical application on the sequential entry game between Walmart and Kmart

Motivating Example

Entry Games between Walmart and Kmart (Jia (2008))

$$U_W = \begin{cases} X'_{W,m}\beta_W - S_{K,m}\delta_W - e_{W,m} & \text{if } S_{W,m} = 1 \\ 0 & \text{if } S_{W,m} = 0 \end{cases}$$

$$U_K = \begin{cases} X'_{K,m}\beta_K - S_{W,m}\delta_K - e_{K,m} & \text{if } S_{K,m} = 1 \\ 0 & \text{if } S_{K,m} = 0 \end{cases}$$

for W (Walmart), K (Kmart), and market m

- $S_{i,m}$: entry decision of player $i \in \{W, K\}$ in market m
- $X_{i,m}$: observable covariates
- $e_{i,m}$: unobservable characteristics (to econometrician)
- δ_W, δ_K: strategic interaction parameters

Motivating Example

Sequential Entry Games (Walmart plays first)

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- Assume $\beta_W = \beta_K = 0$, unique equilibrium
- Stage 2: $P\left(S_{K,m} = 1 | X_m, S_{W,m}\right) = F_{e_K}\left(-S_{W,m}\delta_K\right)$
- Stage 1: $P\left(S_{W,m} = 1 | X_m\right) =$
  $F_{e_W}\left(-P\left(S_{K,m} = 1 | X_m, S_{W,m} = 1\right)\delta_W\right) = F_{e_W}\left(-F_{e_K}\left(-\delta_K\right)\delta_W\right)$
- $P\left(S_{W,m} = 1, S_{K,m} = 1 | X_m\right) = F_{e_K}\left(-\delta_K\right) F_{e_W}\left(-F_{e_K}\left(-\delta_K\right)\delta_W\right)$

Motivating Example

Sequential Entry Games (Kmart plays first)

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Motivating Example

- *"······ the northern regions had been Kmart's backyard until recently, while Wal-Mart started its business from the south and has expertise in serving the southern population."* - Jia (2008)
- The observed joint conditional choice probability is

$$
\begin{aligned}
& P\left(S_{W,m}=1, S_{K,m}=1 | X_m\right) \\
=& \lambda P\left(S_{W,m}=1, S_{K,m}=1 | X_m, \text{Walmart first}\right) \\
& + (1-\lambda) P\left(S_{W,m}=1, S_{K,m}=1 | X_m, \text{Kmart first}\right) \\
=& \lambda F_{e_K}\left(-\delta_K\right) F_{e_W}\left(-F_{e_K}\left(-\delta_K\right)\delta_W\right) \\
& + (1-\lambda) F_{e_W}\left(-\delta_W\right) F_{e_K}\left(-F_{e_W}\left(-\delta_W\right)\delta_K\right)
\end{aligned}
$$

where λ is the proportion of markets in which Walmart moves first

Previous Literature

- Simultaneous Games with Incomplete Information

Seim (2006), Sweeting (2009), Aradillas-lopez (2010), Bajari, Hong, Krainer, and Nekipelov (2010), De Paula and Tang (2012), Wan and Xu (2014), Lewbel and Tang (2015), Aguirregabiria and Mira (2019)

- Sequential Games with Complete / Incomplete Information

Jia (2008), Einav (2010), Blevins (2015), Gayle and Luo (2015), Wagner (2016)

- Nonparametric Estimation of Conditional Moment Models

Ai and Chen (2003), Newey and Powell (2003), Chen and Pouzo (2009, 2012, 2015)

Model

- The game consists of N players: $\{1, \ldots, N\}$
- The action set is $\{a_0, a_1, \ldots, a_L\}$: each player has $L + 1$ actions
- The order of actions is $O = (o(1), \ldots, o(N))$:
  - $o(1) = 1$, $o(2) = 2$ means that player 1 is the first mover and player 2 is the later mover
  - $o(1) = o(2) = 1$ implies a simultaneous game
- The payoff of player i is

$$u_i = \pi_i(s, X, O; \beta_0) + \epsilon_i$$

  where $s = (s_1, \ldots, s_N)$ is the market outcome, $X = (X_1, \ldots, X_N)$ are covariates
- The distribution of ϵ_i is known to players and researchers

Perfect Bayesian Nash Equilibria

A *Perfect Bayesian Nash Equilibrium* (PBNE) is a set of strategies $\left\{S_i\left(\mathcal{J}_{o(i)}, \epsilon_i\right)\right\}_{i=1}^N$ such that for each player i,

$$S_i\left(\mathcal{J}_{o(i)}, \epsilon_i\right) = \arg\max_{a \in \{a_0, \ldots, a_L\}} \Pi_i\left(a, \mathcal{J}_{o(i)}, \epsilon_i\right)$$

- $\Pi_i\left(a, \mathcal{J}_{o(i)}, \epsilon_i\right)$: expected payoff with information set $\mathcal{J}_{o(i)} \equiv \left\{X, O, s_-^{o(i)}\right\}$

$$\begin{aligned}\Pi_i\left(a, \mathcal{J}_{o(i)}, \epsilon_i\right) &= \bar{\pi}_i\left(a, \mathcal{J}_{o(i)}; \beta_0\right) + \epsilon_i \\ &= \sum_{a_+^{o(i)}} \pi_i\left(\left(s_-^{o(i)}, a, a_+^{o(i)}\right), X, O; \beta_0\right) P\left(a_+^{o(i)} \middle| \mathcal{J}_{o(i)}\right) + \epsilon_i\end{aligned}$$

- $s_-^{o(i)}$: actions played before player i
- $a_+^{o(i)}$: actions played after player i

Perfect Bayesian Nash Equilibria: Logit

- If ϵ_i follows an i.i.d. Type-I extreme distribution,

$$P\left(s_i = a \middle| \mathcal{J}_{o(i)}\right) = \frac{\exp\left(\bar{\pi}_i\left(a, \mathcal{J}_{o(i)}; \beta_0\right)\right)}{\sum_{a' \in \{a_0, \ldots, a_L\}} \exp\left(\bar{\pi}_i\left(a', \mathcal{J}_{o(i)}; \beta_0\right)\right)}$$

  and

$$\bar{\pi}_i\left(a, \mathcal{J}_{o(i)}; \beta_0\right) = \sum_{a_+^{o(i)}} \pi_i\left(\left(s_-^{o(i)}, a, a_+^{o(i)}\right), X; \beta_0\right) P\left(a_+^{o(i)} \middle| \mathcal{J}_{o(i)}\right)$$

- By Hotz and Miller (1993), there is one-to-one correspondence between the equilibrium probabilities and expected payoffs
- The equilibrium probabilities $P\left(s_i = a \middle| \mathcal{J}_{o(i)}\right)$ are identified up to the payoff parameters β_0
- The equilibrium may not be unique but finite

Identification with Multiple Equilibria / Order of Actions

- Consider the following example:

$$P(s=\alpha|X) = \sum_{l=1}^{2}\sum_{b=1}^{3}\prod_{i=1}^{N} P^{(b)}\left(s_i=\alpha_i|X, s_{-}^{o_l(i)}, O_l;\beta_0\right)\lambda_{\tau|X,O}(\tau_b)\rho_{O|X}(O_l)$$

if there are 3 PBNE ($b=1,2,3$) and 2 order of actions ($l=1,2$)

- $P^{(b)}(s=\alpha|X,O;\beta_0)$: equilibrium probability of type b
- $\beta_0\in\mathcal{B}\subseteq\mathbb{R}^{d_\beta}$: payoff function parameters
- $\lambda_{\tau|X,O}$: equilibrium selection mechanism
- $\rho_{O|X}$: distribution of the order of actions

Identification Strategy

- It is a special case of the finite mixture model:

$$
\begin{aligned}
& P\left(s=\alpha|X\right) \\
&= \sum_{l=1}^{2}\sum_{b=1}^{3}\prod_{i=1}^{N} P^{(b)}\left(s_i=\alpha_i|X, s_{-}^{o_l(i)}, O_l; \beta_0\right) \lambda_{\tau|X,O}\left(\tau_b\right)\rho_{O|X}\left(O_l\right) \\
&= \sum_{k=1}^{6} P^{(k)}\left(s=\alpha|X, O_k; \beta_0\right) h_{0,k}\left(X\right)
\end{aligned}
$$

  such that $\sum_{k=1}^{6} h_{0,k}\left(X\right)=1$
- $P\left(s=\alpha|X\right)$ is identified from data
- $P^{(k)}\left(s=\alpha|X, O_k; \beta_0\right)$ is identified up to β_0 by model
- The **mixture weights** $h_{0,k}\left(X\right)$ are nonparametric functions of X

Identification of $\lambda_{\tau|X,O}$ and $\rho_{O|X}$

- Suppose $\{\beta_0, h_{0,1}(X), \ldots, h_{0,6}(X)\}$ is identified
- For the previous example with 3 PBNE ($b = 1, 2, 3$) and 2 order of actions ($l = 1, 2$)

$$
\begin{aligned}
h_{0,1}(X) &= \lambda_{\tau|X,O}(\tau_1)\rho_{O|X}(O_1) \\
h_{0,2}(X) &= \lambda_{\tau|X,O}(\tau_2)\rho_{O|X}(O_1) \\
h_{0,3}(X) &= \lambda_{\tau|X,O}(\tau_3)\rho_{O|X}(O_1)
\end{aligned}
$$

implies that

$$
\rho_{O|X}(O_1) = \sum_{k=1}^{3} h_{0,k}(X), \quad \lambda_{\tau|X,O}(\tau_1) = \frac{h_{0,1}(X)}{\sum_{k=1}^{3} h_{0,k}(X)}
$$

Identification of Structural Parameters

- The structural parameters $\left\{\beta_0, \lambda_{\tau|X,O}, \rho_{O|X}\right\}$ are **identified** if there is a unique solution $\left\{\beta_0, \left\{h_{0,k}\left(X\right)\right\}_{k=1}^{N_\kappa}\right\}$ of the equations

$$P\left(s=\alpha|X\right)=\sum_{k=1}^{N_\kappa} P^{(k)}\left(s=\alpha|X,O_k;\beta_0\right)h_{0,k}\left(X\right)$$

  for $X-$a.s. and all joint actions α
- N_κ is the number of types for unobserved heterogeneity (τ and O)

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$$P(s = \alpha|X) = \sum_{k=1}^{N_\kappa} P^{(k)}(s = \alpha|X, O_k; \beta_0)\, h_{0,k}(X)$$

  for $X-$a.s. and all joint actions α
- N_κ is the number of types for unobserved heterogeneity (τ and O)

1. **Sufficient Rank condition**: (by Rouché–Capelli) the solution of the linear system is unique only at $\beta_0 \in \mathcal{B}$
2. **Necessary Order condition**: $(L + 1)^N \geq N_\kappa$

Remarks

- The maximum number of N_κ depends on the number of players N and the number of actions L
- The possible orders of actions O increase in N
  - 3 orders for two players: $O = (1, 2), (2, 1), (1, 1)$
  - 13 orders for three players: $O = (1, 1, 1), (1, 1, 2) \ldots, (1, 2, 3)$
  - 75 orders for four players: $O = (1, 1, 1, 1), \ldots, (1, 2, 3, 4)$
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  - $(L + 1)^N$ joint actions for N players

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- The possible joint actions increase in L
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- The *exclusion restriction* is helpful to attain the order condition: there exists $X_s \subseteq X$ s.t. $\rho_{O|X}(\cdot|X) = \rho_{O|X}(\cdot|X_s)$
- $X_v = X \backslash X_s$ are instruments: $P(s = \alpha|X)$ varies with X_v but $\rho_{O|X}(\cdot|X)$ does not

Conditional Moment Restrictions

- Observe $Z^m = (s^m, X^m)$ for $m = 1, \ldots, n$
- Parameter of interest: $\theta_0 = \left(\beta_0, h_{0,1}, \ldots, h_{0,N_\kappa}\right) \in \Theta = \mathcal{B} \times \mathcal{H}$
- Conditional moments:

$$E\left[\ell_j\left(Z^m, \theta_0\right) | X^m\right] = 0$$

$$E\left[1 - \sum_{k=1}^{N_\kappa} h_{0,k}\left(X^m\right) \middle| X^m\right] = 0$$

where

$$\ell_j\left(Z^m, \theta\right) = 1\left\{s^m = \alpha^j\right\} - \sum_{k=1}^{N_\kappa} P^{(k)}\left(s^m = \alpha^j | X^m, O_k; \beta\right) h_k\left(X^m\right)$$

for $j = 1, \ldots, (L+1)^N - 1$

The Sieve Minimum Distance Estimator

- The SMD estimator $\hat{\theta}_n = \left(\hat{\beta}_n, \hat{h}_{1,n}, \ldots, \hat{h}_{N_\kappa,n}\right) \in \Theta_n = \mathcal{B} \times \mathcal{H}_n$ solves

$$\min_{\theta \in \mathcal{B} \times \mathcal{H}_n} \mathcal{G}_n \left(Z^m, \theta\right)' \left(I \otimes \left(P^{J_n'} P^{J_n}\right)^{-1}\right) \mathcal{G}_n \left(Z^m, \theta\right)$$

  where

$$\mathcal{G}_n \left(Z^m, \theta\right) = \sum_{m=1}^{n} \ell \left(Z^m, \theta\right) \otimes p^{J_n} \left(X^m\right)$$

$$p^{J_n} \left(X\right) = \left(p_1 \left(X\right), \ldots, p_{J_n} \left(X\right)\right)' \text{ (sieve basis)}$$

$$P^{J_n} = \left(p^{J_n}\left(X^1\right), \ldots, p^{J_n}\left(X^n\right)\right)' \text{ (sieve basis)}$$

- $\hat{h}_{k,n}(x) = \sum_{j=1}^{K_n} c_j p_j(x) \in \mathcal{H}_{k,n}$ and $\mathcal{H}_n = \prod_{k=1}^{N_\kappa} \mathcal{H}_{k,n}$

Asymptotic Properties of the Estimator

- Assumptions for Consistency

1. $\{Z^m\}_{m=1}^n$ i.i.d., $\mathcal{X} \subseteq \mathbb{R}^{d_x}$ compact, f_X bounded
2. CCPs $P(s = \alpha | X)$ and $h_{0,k}(X)$ are smooth functions (Hölder space with $\gamma > d_x/2$)
3. Equilibrium probability $\left\{P^{(k)}\left(s_i = \alpha_i | \mathcal{J}_{o_k(i)}; \beta\right)\right\}_{k=1}^{N_\kappa}$ is continuously differentiable on $\mathcal{X}$ and $\mathcal{B}$
4. Smoothing parameters satisfy $(L+1)^N J_n \geq d_\beta + N_\kappa K_n$, $K_n \to \infty$, $J_n/n \to 0$ as $n \to \infty$

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- **Consistency**: $\left\|\hat{\theta}_n - \theta_0\right\|_s = o_p(1)$ where

$$\|\theta\|_s = \|\beta\|_E + \max_{k=1,\ldots,N_\kappa} \sup_{x \in \mathcal{X}} |h_k(x)|$$

Asymptotic Properties of the Estimator

- Define the sieve variance components:

$$D_n = E\left[\Delta_n(X)' \Delta_n(X)\right]$$
$$\Psi_n = E\left[\Delta_n(X)' \ell(Z, \theta_0) \ell(Z, \theta_0)' \Delta_n(X)\right]$$

where

$$\Delta_n(X) \equiv \left[\Delta_{\beta_0}(X), \Delta_{h_0,n}(X)\right]$$
$$\Delta_{\beta_0}(X) \equiv \sum_{k=1}^{N_\kappa} \frac{\partial}{\partial \beta'} P^{(k)}(X, O; \beta_0)\, h_{0,k}(X)$$
$$\Delta_{h_0,n}(X) \equiv \left[P^{(1)}(X, O; \beta_0) \otimes p^{K_n'}(X), \ldots, P^{(N_\kappa)}(X, O; \beta_0) \otimes p^{K_n'}(X)\right]$$

- Assume D_n and $E\left[\ell(Z, \theta_0) \ell(Z, \theta_0)' | X\right]$ are positive definite

Asymptotic Properties of the Estimator

- (Additional) Assumptions for Asymptotic Normality

1. Equilibrium probability $\left\{P^{(k)}\left(s_i=\alpha_i|\mathcal{J}_{o_k(i)};\beta\right)\right\}_{k=1}^{N_\kappa}$ is twice continuously differentiable on $\mathcal{X}$ and $\mathcal{B}$
2. $J_n^{-\gamma/d_x}=o\left(n^{-1/4}\right)$, $K_n^{-\gamma/d_x}=o\left(n^{-1/4}\right)$, and $J_nK_n\log n=o\left(n^{1/2}\right)$

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- **Asymptotic Normality**:

$$\sqrt{n}\left\|v_n^*\right\|_{sd,\beta}^{-1}\left(\hat{\beta}_n - \beta_0\right) \xrightarrow{d} N\left(0, I_{d_\beta}\right)$$

where

$$\left\|v_n^*\right\|_{sd,\beta}^2 = G_\beta' D_n^{-1} \Psi_n D_n^{-1} G_\beta$$

$$G_\beta \equiv \left[I_{d_\beta}, \mathbb{O}_{d_\beta \times (N_\kappa - 1) K_n}\right]'$$

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- **Asymptotic Normality**:

$$\sqrt{n}\left\|v_n^*\right\|_{sd,h}^{-1}\left(\hat{h}_{k,n}(x)-h_{0,k}(x)\right)\xrightarrow{d}N(0,1)$$

where

$$\begin{aligned}\left\|v_n^*\right\|_{sd,h}^2&=G_{h_k}'D_n^{-1}\Psi_nD_n^{-1}G_{h_k}\\ G_{h_k}&=\left[\mathbb{O}_{1\times\left(d_\beta+(k-1)K_n\right)},p^{K_n'}(x),\mathbb{O}_{1\times(N_\kappa-k)K_n}\right]'\end{aligned}$$

Simulation Setup I

$$u_1 = \begin{cases} X_1\beta_1 + \Delta_1 S_2 - \epsilon_1 & \text{if } S_1 = 1 \\ 0 & \text{if } S_1 = 0 \end{cases}$$

$$u_2 = \begin{cases} X_2\beta_2 + \Delta_2 S_1 - \epsilon_2 & \text{if } S_2 = 1 \\ 0 & \text{if } S_2 = 0, \end{cases}$$

- Parameters: $(\beta_1, \beta_2) = (3, 2)$, $(\Delta_1, \Delta_2) = (-6, -4)$
- ϵ_1, ϵ_2: i.i.d. Type-I Extreme Distributions
- # of Games: 500, 1000, 5000
- *Experiment 1*: $X_i \in \{0.5, 1.0, 1.5, 2.0\}$ uniformly distributed, sequential actions only

$$P\left(O = (1,2)' \,|X\right) = \frac{X_1 + X_2}{10}$$

Simulation Results: Experiment I

	$n = 500$	$n = 1000$	$n = 5000$
β_1	3.0224	2.9317	2.9851
	(1.0244)	(0.6819)	(0.2848)
β_2	3.2107	2.5848	2.1032
	(1.6446)	(0.9277)	(0.2727)
Δ_1	-6.0939	-5.8906	-5.9763
	(1.9919)	(1.3065)	(0.5412)
Δ_2	-6.4342	-5.1746	-4.2078
	(3.3420)	(1.8622)	(0.5303)
$P\left(O = (1,2)' \mid X = (1,1)\right)$	0.2477	0.2351	0.2066
	(0.1061)	(0.0810)	(0.0422)
$P\left(O = (1,2)' \mid X = (2,2)\right)$	0.4482	0.4314	0.4075
	(0.1593)	(0.1241)	(0.0567)
$P\left(O = (2,1)' \mid X = (1,2)\right)$	0.3680	0.3350	0.3084
	(0.1992)	(0.1384)	(0.0507)

Simulation Setup II

$$u_1 = \begin{cases} X_1\beta_1 + \Delta_1 S_2 - \epsilon_1 & \text{if } S_1 = 1 \\ 0 & \text{if } S_1 = 0 \end{cases}$$

$$u_2 = \begin{cases} X_2\beta_2 + \Delta_2 S_1 - \epsilon_2 & \text{if } S_2 = 1 \\ 0 & \text{if } S_2 = 0, \end{cases}$$

- Parameters: $(\beta_1, \beta_2) = (3, 2)$, $(\Delta_1, \Delta_2) = (-6, -4)$
- ϵ_1, ϵ_2: i.i.d. Type-I Extreme Distributions
- # of Games: 500, 1000, 5000
- *Experiment 2*: $X_i \in \{1, 2, 3, 4\}$ uniformly distributed, simultaneous + sequential actions

$$P\left(O = (1,2)' \,|X\right) = X_1/10$$
$$P\left(O = (2,1)' \,|X\right) = X_2/10$$

Simulation Results: Experiment II

	$n = 500$	$n = 1000$	$n = 5000$
β_1	4.0194	3.4223	3.2314
	(1.7968)	(1.1530)	(0.5008)
β_2	2.8799	2.3853	2.1518
	(1.6547)	(1.0219)	(0.3570)
Δ_1	-8.0242	-6.8062	-6.4407
	(3.8142)	(2.4815)	(1.0980)
Δ_2	-5.8660	-4.7713	-4.3089
	(3.5192)	(2.2413)	(0.7824)
$P\left(O = (1,2)' \,\vert X = (1,2)\right)$	0.1423	0.1206	0.0995
	(0.1742)	(0.1187)	(0.0293)
$P\left(O = (2,1)' \,\vert X = (2,1)\right)$	0.1639	0.1310	0.1050
	(0.2082)	(0.1344)	(0.0413)
$P\left(O = (2,1)' \,\vert X = (2,2)\right)$	0.2196	0.2164	0.1998
	(0.1604)	(0.1394)	(0.0487)

Estimation Under the Misspecified Order of Actions

Simultaneous (Correct)	$n = 500$	$n = 1000$	$n = 5000$
β_1	3.6114	3.4164	3.1269
	(2.6312)	(2.2967)	(0.9769)
β_2	2.4729	2.3835	2.1512
	(1.6111)	(1.4545)	(0.6344)
Δ_1	-5.9525	-5.4472	-5.9944
	(4.4415)	(3.7776)	(1.7528)
Δ_2	-4.3551	-3.5876	-3.9883
	(3.3789)	(2.5196)	(1.1988)

- DGP: Simultaneous Actions
- Estimation: Simultaneous Actions
- Parameters: $(\beta_1, \beta_2) = (3, 2)$, $(\Delta_1, \Delta_2) = (-6, -4)$

Estimation Under the Misspecified Order of Actions

Simultaneous (Misspecified)	$n = 500$	$n = 1000$	$n = 5000$
β_1	1.9210	1.9710	2.0053
	(0.8322)	(0.4596)	(0.1838)
β_2	1.1123	0.9349	0.8546
	(0.6155)	(0.4636)	(0.3436)
Δ_1	-2.8356	-2.9900	-3.0692
	(1.4094)	(0.8617)	(0.1795)
Δ_2	-1.5654	-1.1462	-0.9185
	(1.4947)	(1.1220)	(0.4424)

- DGP: Simultaneous Actions
- Estimation: Sequential Actions
- Parameters: $(\beta_1, \beta_2) = (3, 2)$, $(\Delta_1, \Delta_2) = (-6, -4)$

Entry Game of Walmart vs Kmart

$$u_i = \begin{cases} X_i'\beta_i + S_{-i}\Delta_i - \epsilon_i & \text{if } S_i = 1 \\ 0 & \text{if } S_i = 0 \end{cases}$$

- S_i: entry decision of player $i \in \{W, K\}$
- X_i: population, retail sales, urban population (%), # of small stores, # of nearby branches
- ϵ_i: private information with i.i.d. Type-I extreme distribution
- Δ_i: Interaction effects
- Jia (2008), Competition of Walmart and Kmart
  - Sample Size: 2065 counties
  - Entry of Walmart: 48% / Entry of Kmart: 19% (1997)
- Sequential Actions (Walmart $\longrightarrow$ Kmart or Kmart $\longrightarrow$ Walmart) + Simultaneous Actions
- Order distribution is a function of regional dummies

Entry Game Estimation (Simultaneous Actions Only)

	Walmart	Kmart
Log Population	2.0205	2.0422
	(0.2051)	(0.3739)
Log Retail Sales	0.9744	2.1261
per Capita	(0.1625)	(0.3034)
Percentage of	1.0858	0.9940
Urban Population	(0.2701)	(0.4312)
Small Stores	-0.0277	-0.0362
	(0.0270)	(0.0245)
# of nearby branches	-3.4444	-0.6699
	(1.1099)	(0.9996)
Constant	-13.5967	-24.9788
	(1.5686)	(3.0105)
Interaction Effects	-0.9934	-0.6866
(Δ_W, Δ_K)	(0.5338)	(1.3232)

Entry Game Estimation (Sequential Actions Only)

	Walmart	Kmart
Log Population	2.6290	2.1567
	(0.2243)	(0.2062)
Log Retail Sales	1.4394	1.9640
per Capita	(0.1856)	(0.2599)
Percentage of	1.2483	1.0260
Urban Population	(0.3337)	(0.3533)
Small Stores	-0.0724	-0.0298
	(0.0325)	(0.0241)
# of nearby branches	-3.7524	-1.4352
	(0.9668)	(0.9091)
Constant	-18.8790	-23.2951
	(1.6805)	(2.4164)
Interaction Effects	-1.8410	-1.4281
(Δ_W, Δ_K)	(0.2543)	(0.2245)

Estimated Order of Actions (Sequential Actions Only)

	Midwest	South	Other Regions
Walmart $\longrightarrow$ Kmart	0.0827 (0.1372)	0.9999 (0.0000)	0.0011 (0.0171)
Kmart $\longrightarrow$ Walmart	0.9173 (0.1372)	0.0001 (0.0000)	0.9989 (0.0171)

Entry Game Estimation (Simultaneous + Sequential)

	Walmart	Kmart
Log Population	2.5840	2.5544
	(0.2981)	(0.3393)
Log Retail Sales	1.4665	2.1796
per Capita	(0.2187)	(0.3094)
Percentage of	1.2476	1.1590
Urban Population	(0.3808)	(0.3936)
Small Stores	-0.0757	-0.0366
	(0.0352)	(0.0275)
# of nearby branches	-3.3114	-1.6872
	(0.9876)	(1.0192)
Constant	-18.9547	-25.6431
	(2.2993)	(3.1450)
Interaction Effects	-2.2053	-2.6667
(Δ_W, Δ_K)	(0.6367)	(0.5309)

Estimated Order of Actions (Simultaneous + Sequential)

	Midwest	South	Other Regions
Walmart → Kmart	0.0018 (0.0158)	0.2449 (0.2006)	0.0450 (0.0960)
Kmart → Walmart	0.4705 (0.1408)	0.0042 (0.0606)	0.6836 (0.1956)
Simultaneous	0.5278 (0.1390)	0.7509 (0.2051)	0.2714 (0.1956)

Conclusion

- This paper studies the identification and estimation of incomplete information sequential games considering multiple PBNE and unknown order of actions
- There are some necessary and sufficient conditions to identify structural parameters, and an exclusion restriction is helpful to attain the necessary order condition for identification
- The SMD estimator based on conditional moments is consistent and asymptotically normal
- Monte Carlo simulation & empirical application to the Walmart vs. Kmart entry game highlight the importance of correct specification on the order of actions